

DEGREES OF REGULAR SEQUENCES WITH A SYMMETRIC GROUP ACTION

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ABSTRACT. We consider ideals in a polynomial ring that are generated by regular sequences of homogeneous polynomials and are stable under the action of the symmetric group permuting the variables. In previous work, we determined the possible isomorphism types for these ideals. Following up on that work, we now analyze the possible degrees of the elements in such regular sequences. For each case of our classification, we provide some criteria guaranteeing the existence of regular sequences in certain degrees.

1. INTRODUCTION

Consider the graded polynomial ring $R = \mathbb{C}[x_1, x_2, \dots, x_n]$. A set of n homogeneous polynomials f_1, f_2, \dots, f_n is a maximal regular sequence in R if the only common zero of these n polynomials is the point $(0, 0, \dots, 0)$. A sequence g_1, g_2, \dots, g_t is a regular sequence in R if it can be extended to a maximal regular sequence in R .

We suppose that G is a group acting linearly on R via an action which preserves the grading. The subring $R^G := \{f \in R : \forall \sigma \in G, \sigma \cdot f = f\}$ is called the ring of invariants. There has been some interest in determining the degrees (d_1, d_2, \dots, d_t) for which there exists a regular sequence in R^G with $\deg(f_i) = d_i$. Dixmier [6] made a conjecture concerning this question for the classical case of the action of $\mathrm{SL}(2, \mathbb{C})$ on an irreducible representation. This conjecture has attracted some attention. See for example [17, 7, 1]. Recently, a few authors have taken up this question for the natural action of the symmetric group on R . See [5, 4, 14].

We consider a more general question. Our goal is to determine the degrees of a maximal regular sequence f_1, f_2, \dots, f_n in R such that the ideal $I := (f_1, f_2, \dots, f_n)$ is stable under the group action. This is equivalent to the artinian quotient algebra R/I inheriting the action of the group.

We will also restrict our attention to the natural action of the symmetric group \mathfrak{S}_n permuting the variables. In our earlier paper [10], it is shown that there are four possible representation types for the action of \mathfrak{S}_n on I (the notation follows that of [19]):

- (1) the trivial representation $S^{(n)}$, given by all f_i being symmetric polynomials;
- (2) the alternating representation $S^{(1^n)}$, given by one alternating polynomial, together with up to $n - 1$ symmetric polynomials;
- (3) the standard representation $S^{(n-1,1)}$, possibly together with one symmetric polynomial;

- (4) the representation $S^{(2,2)}$, together with up to two symmetric polynomials (this only occurs when $n = 4$).

Our earlier paper shows examples of regular sequences corresponding to all four cases, but does not address the question of how “often” such regular sequences can appear or, more precisely, in what degrees they can be realized. Here we give explicit answers showing in which degrees it is possible to find a regular sequence for each of the above four representation types for $n \leq 4$. We also derive a number of results for general values of n .

Note also that our results relating to the first case above, actually apply to the degrees of regular sequences of homogeneous polynomials in the polynomial ring $\mathbb{C}[y_1, \dots, y_n]$, with the non-standard grading given by $\deg(y_i) = i$. This case corresponds geometrically to the homogeneous coordinate ring of a weighted projective space.

2. REGULAR SEQUENCES OF SYMMETRIC POLYNOMIALS

We consider the polynomial ring $R = \mathbb{C}[x_1, x_2, \dots, x_n]$ in n indeterminates equipped with the standard grading. The symmetric group \mathfrak{S}_n acts naturally on R by permuting the variables. It is well known that the invariant subring $R^{\mathfrak{S}_n}$ can be identified with the subalgebra $\mathbb{C}[e_1, e_2, \dots, e_n]$ generated by the elementary symmetric polynomials [11]. In particular, $R^{\mathfrak{S}_n}$ is a polynomial ring equipped with the non-standard grading $\deg(e_i) = i$.

2.1. Degree sequences. We are concerned with the degrees of elements of homogeneous regular sequences in $R^{\mathfrak{S}_n}$.

Definition 1. Let (d_1, d_2, \dots, d_n) be an (unordered) sequence of n positive integers. If there exists a homogeneous regular sequence $f_1, f_2, \dots, f_n \in R^{\mathfrak{S}_n}$ with $\deg(f_i) = d_i$ then we say that (d_1, d_2, \dots, d_n) is a *regular degree sequence*.

Proposition 2. Suppose (d_1, d_2, \dots, d_n) is a regular degree sequence. For $i = 2, 3, \dots, n$ we define $\beta_i := \#\{1 \leq j \leq n : i \mid d_j\}$. Then

$$(*) \quad \beta_i \geq \left\lfloor \frac{n}{i} \right\rfloor \quad \text{for all } i = 1, 2, \dots, n$$

In particular, $n! \mid \prod_{j=1}^n d_j$.

Proof. If (d_1, d_2, \dots, d_n) is a regular degree sequence, then there exists a homogeneous regular sequence f_1, f_2, \dots, f_n in $R^{\mathfrak{S}_n}$ with $\deg(f_i) = d_i$. The graded subring $A = \mathbb{C}[f_1, f_2, \dots, f_n]$ is a polynomial ring and $R^{\mathfrak{S}_n}$ is a free A -module: $R^{\mathfrak{S}_n} \cong \bigoplus_{\gamma \in \Gamma} A \cdot \gamma$ for some set of homogeneous elements $\Gamma \subset R^{\mathfrak{S}_n}$ [3, Lemma 6.4.13]. Thus the Hilbert series of $R^{\mathfrak{S}_n}$ and A are related by

$$\mathcal{H}(R^{\mathfrak{S}_n}, t) = \sum_{\gamma \in \Gamma} t^{\deg(\gamma)} \mathcal{H}(A, t).$$

Since $\mathcal{H}(R^{\mathfrak{S}_n}, t) = \prod_{i=1}^n (1 - t^i)^{-1}$ and $\mathcal{H}(A, t) = \prod_{i=1}^n (1 - t^{d_i})^{-1}$, we see that

$$\prod_{i=1}^n \frac{1 - t^{d_i}}{1 - t^i} = \sum_{\gamma \in \Gamma} t^{\deg(\gamma)}$$

is a non-negative integer polynomial.

Working over \mathbb{Q} , all the irreducible factors of $(1 - t^d)$ are cyclotomic polynomials. Specifically $(1 - t^d) = \prod_{i|d} \Phi_i(t)$, where Φ_i denotes the i^{th} cyclotomic polynomial. Since $\#\{1 \leq j \leq n : i \mid j\} = \lfloor n/i \rfloor$, we see that $\prod_{i=1}^n \frac{1-t^{d_i}}{1-t^i}$ is an integer polynomial if and only if $\beta_i \geq \lfloor n/i \rfloor$ for all $i = 1, 2, \dots, n$.

To prove the final assertion, we cancel the factors of $(1 - t)$ from the numerator and denominator. Thus

$$\prod_{i=1}^n \frac{1 + t + t^2 + \dots + t^{d_i}}{1 + t + t^2 + \dots + t^i} = \sum_{\gamma \in \Gamma} t^{\deg(\gamma)}.$$

Evaluating at $t = 1$ we see that $(\prod_{i=1}^n d_i) / n! = |\Gamma| = \text{rank of } R^{\mathfrak{S}_n} \text{ as an } A\text{-module.}$ \square

Remark 3. Our inequality (*) was first observed by Conca, Krattenthaler and Watanabe for regular sequences of power sums [5, Lemma 2.6 (2)]. The three authors also showed that the product of the d_i is divisible by $n!$ [5, Lemma 2.8]. This seems to be the first time the restriction that

$$\prod_{i=1}^n \frac{1 - t^{d_i}}{1 - t^i} = \sum_{\gamma \in \Gamma} t^{\deg(\gamma)}$$

is a non-negative integer polynomial has been observed.

Suppose (d_1, d_2, \dots, d_n) is a regular degree sequence. Since $\oplus_{d \leq i} R_d^{\mathfrak{S}_n} \subset \mathbb{C}[e_1, e_2, \dots, e_i]$ and hence cannot contain a regular sequence with more than i terms, we deduce that (d_1, d_2, \dots, d_n) must also satisfy the following condition

$$(\dagger) \quad \#\{j : d_j \leq i\} \leq i \quad \text{for all } i = 1, 2, \dots, n.$$

Definition 4. Let (d_1, d_2, \dots, d_n) be an (unordered) sequence of n positive integers. We say that (d_1, d_2, \dots, d_n) is *permissible* if it satisfies the two conditions (*) and (\dagger). Thus every regular degree sequence is permissible.

Note that if there exists a *matching*, i.e., a permutation $\pi \in \mathfrak{S}_n$ such that i divides $d_{\pi(i)}$ for all $i = 1, 2, \dots, n$ then (d_1, d_2, \dots, d_n) is a regular degree sequence as is shown by the regular sequence of polynomials $(e_i)^{d_{\pi(i)}/i}$ for $i = 1, 2, \dots, n$.

2.2. Regular degree sequences for $n \leq 4$.

Theorem 5.

- (1) For $n = 2$, a degree sequence is regular if and only if it is permissible if and only if it satisfies (*).
- (2) For $n = 3$, a degree sequence is regular if and only if it is permissible.
- (3) For $n = 4$, every permissible degree sequence except $(1, 2, 5, 12\delta)$, $(2, 2, 5, 12\delta)$ and $(5, 2, 5, 12\delta)$ is regular.

Proof. (1) If (d_1, d_2) satisfies (*) then at least one of d_1 or d_2 is even and so we have a matching.

(2) Let $n = 3$ and suppose that (d_1, d_2, d_3) is permissible but has no matching. Then, without loss of generality, 6 divides d_3 while d_1 and d_2 are both odd numbers not divisible by 3 with $d_2 \geq d_1$. Now condition (\dagger) implies that $d_2 \geq 2$ and thus $d_2 \geq 5$. Thus we have a regular sequence $e_1^{d_1}, e_3 e_2^{(d_2-3)/2}, (e_2^3 + e_3^2)^{d_3/6}$ with degrees (d_1, d_2, d_3) .

(3) Let $n = 4$ and suppose that (d_1, d_2, d_3, d_4) is permissible but has no matching. Note that $R_1^{\mathfrak{S}_n} \oplus R_2^{\mathfrak{S}_n} \oplus R_5^{\mathfrak{S}_n}$ is contained in the ideal generated by e_1 and e_2 . This implies that a regular degree sequence must satisfy $\#\{i : d_i \in \{1, 2, 5\}\} \leq 2$. This shows that the three permissible degree sequences $(1, 5, 2, 12\delta)$, $(2, 5, 2, 12\delta)$, $(5, 5, 2, 12\delta)$ are not regular.

Condition $(*)$ implies that two of the d_i are even, one is divisible by 3 and one is divisible by 4. Without loss of generality, d_3 and d_4 are both even, $d_4 = 4\delta$ is divisible by 4 and $d_2 \geq d_1$. Since there is no matching, neither d_1 nor d_2 is divisible by 3. Thus either d_3 is a multiple of 6 or d_4 is a multiple of 12.

First we consider the case where $d_3 = 6\beta$ is a multiple of 6. Since there is no matching both d_1 and d_2 are odd integers not divisible by 3. Thus (\dagger) implies that $d_2 \geq 5$. Therefore $e_1^{d_1}, e_3 e_2^{(d_2-3)/2}, (e_2^3 + e_3^2)^\beta, e_4^\delta$ is a regular sequence of the required degrees.

Thus we may suppose that $d_4 = 12\delta$ is a multiple of 12. Now we adjust our labelling as follows. We suppose that d_3 is the largest of those elements of $\{d_1, d_2, d_3\}$ which are even. Further we assume that $d_2 \geq d_1$. Furthermore, since there is no matching, 3 divides neither d_1 nor d_2 .

Since $d_2 \geq 2$ we may write $d_2 = 2p + 3q$ where p and q are non-negative integers. Suppose first that $d_3 \geq 4$ and define

$$f := \begin{cases} e_4^{d_3/4}, & \text{if } d_3 \equiv 0 \pmod{4}; \\ e_4^{(d_3-6)/4} (e_2^3 + e_3^2), & \text{if } d_3 \equiv 2 \pmod{4}. \end{cases}$$

Then $e_1^{d_1}, e_2^p e_3^q, f, (e_2^6 + e_3^4 + e_4^3)^\delta$ is a regular sequence of degrees (d_1, d_2, d_3, d_4) .

Finally we suppose that $d_3 = 2$. Then either $d_2 = 2$ or d_2 is odd. But we have seen that $(1, 2, 2, 12\delta)$ and $(2, 2, 2, 12\delta)$ are not regular degree sequences and thus d_2 must be odd. Since 3 does not divide d_2 , we have $d_2 \geq 5$. If $d_2 = 5$ then $d_1 \in \{1, 2, 5\}$ which is again not possible since $(1, 5, 2, 12\delta)$, $(2, 5, 2, 12\delta)$ and $(5, 5, 2, 12\delta)$ are not regular degree sequences. Therefore $d_2 \geq 7$ if $d_3 = 2$. Thus we may write $d_2 = 3p + 4q$. Then $e_1^{d_1}, e_3^p e_4^q, e_2, (e_3^4 + e_4^3)^\delta$ is a regular sequence of the required degrees. \square

Table 1 summarizes the regular sequences we have found when $n = 4$. The first row corresponds to a matching.

Note that we have in fact proved the following result.

Corollary 6. Suppose that d_2, d_3, d_4 are three positive integers such that $4 \mid d_4$, d_3 is even, and $3 \mid d_2 d_3 d_4$. Then there exist three symmetric polynomials f_2, f_3, f_4 (as given in Table 1) of degrees d_2, d_3, d_4 respectively such that e_1, f_2, f_3, f_4 is a regular sequence.

Degrees				Symmetric Polynomials			
d_1	d_2	d_3	d_4	f_1	f_2	f_3	f_4
d_1	3β	2γ	4δ	$e_1^{d_1}$	e_3^β	e_2^γ	e_4^δ
d_1	$d_2 \geq 5$	6γ	4δ	$e_1^{d_1}$	$e_3 e_2^{(d_2-3)/2}$	$(e_2^3 + e_3^2)^\gamma$	e_4^δ
d_1	$d_2 \geq 2$	4β	12δ	$e_1^{d_1}$	$e_2^p e_3^q$	e_4^β	$(e_2^6 + e_3^4 + e_4^3)^\delta$
d_1	$d_2 \geq 2$	$4\beta + 2 \geq 6$	12δ	$e_1^{d_1}$	$e_2^p e_3^q$	$(e_2^3 + e_3^2) e_4^{\beta-1}$	$(e_2^6 + e_3^4 + e_4^3)^\delta$
d_1	$d_2 \geq 7$	2	12δ	$e_1^{d_1}$	$e_3^p e_4^q$	e_2	$(e_3^4 + e_4^3)^\delta$

TABLE 1. Regular sequences of symmetric polynomials for $n = 4$

Remark 7. Note that the degree sequence $(2, 5, 2, 12)$ (which is not regular) has the property that

$$\frac{(1-t^2)(1-t^5)(1-t^2)(1-t^{12})}{(1-t)(1-t^2)(1-t^3)(1-t^4)} = 1 + t + t^3 + 2t^4 + 2t^7 + t^8 + t^{10} + t^{11}$$

is a non-negative integer polynomial.

For larger values of n little is known. The following statement was proved in [5, Prop. 2.9] using sequences of power sums and homogeneous symmetric polynomials.

Proposition 8. For every positive integer a the sequence of consecutive degrees $(a, a+1, a+2, \dots, a+n-1)$ is a regular degree sequence.

2.3. Regular sequences with an alternating polynomial. A polynomial $f \in R$ is said to be *alternating* if, for all $\sigma \in \mathfrak{S}_n$, $\sigma f = \pm f$, depending on the sign of σ . As an example, the Vandermonde determinant

$$\Delta := \det \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i) \in R$$

is clearly alternating. In fact, every homogeneous alternating polynomial in R is divisible by Δ , the quotient being a homogeneous symmetric polynomial.

As noted in [10, Prop. 2.5], there exist homogeneous regular sequences $f_1, f_2, \dots, f_t, g\Delta$ in R with f_1, f_2, \dots, f_t and g symmetric polynomials. These sequences are closely related to sequences of symmetric polynomials.

Lemma 9. Let $f_1, f_2, \dots, f_t, g, h \in R$ be homogeneous polynomials. Then the sequence f_1, f_2, \dots, f_t, gh is regular if and only if both f_1, f_2, \dots, f_t, g and f_1, f_2, \dots, f_t, h are regular.

Proof. Suppose f_1, f_2, \dots, f_t form a regular sequence. Then gh is not a zero-divisor modulo (f_1, f_2, \dots, f_t) if and only if both g and h are not zero-divisors modulo (f_1, f_2, \dots, f_t) . \square

The following is an immediate consequence of Lemma 9.

Proposition 10. Let $f_1, f_2, \dots, f_t, g \in R$ be homogeneous symmetric polynomials. The sequence $f_1, f_2, \dots, f_t, g\Delta$ is regular if and only if both f_1, f_2, \dots, f_t, g and $f_1, f_2, \dots, f_t, \Delta$ are regular.

Proposition 10 allows to rule out existence of regular sequences of certain degrees that contain an alternating polynomial.

Example 11. For $n = 4$, Δ has degree 6. By Theorem 5 (3), there is no regular sequence of homogeneous symmetric polynomials f_1, f_2, f_3, g of degrees 1, 2, 5, 12δ . Therefore, Proposition 10 implies there is no regular sequence $f_1, f_2, f_3, g\Delta$ of degrees 1, 2, 5, $12\delta + 6$.

Remark 12. The polynomial Δ^{2k} is symmetric for all positive integers k . Moreover, the sequence $f_1, f_2, \dots, f_t, \Delta$ is regular if and only if $f_1, f_2, \dots, f_t, \Delta^{2k}$ is regular (cf. [8, Cor. 17.8 a]). As a consequence, we can exclude the existence of regular sequences in certain degrees. For example, there is no regular sequence of homogeneous polynomials f_1, f_2, f_3, Δ with f_1, f_2, f_3 symmetric of degrees 1, 2, 5 because f_1, f_2, f_3, Δ^2 would violate Theorem 5 (3).

3. REGULAR SEQUENCES AND THE STANDARD REPRESENTATION

We begin this section by recalling some basic facts about the representation theory of the symmetric group \mathfrak{S}_n over a field of characteristic zero. We refer the reader to [19, Ch. 2] for the details.

We write $\lambda \vdash a$ to denote that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition of the integer a , i.e., that $\lambda_1 + \lambda_2 + \dots + \lambda_r = a$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$. The irreducible representations of \mathfrak{S}_n are in bijection with the partitions of n ; for $\lambda \vdash n$, we denote by S^λ the corresponding irreducible. Every finite dimensional representation of \mathfrak{S}_n decomposes into a direct sum of copies of the S^λ .

The irreducible representation $S^{(n-1,1)}$ of \mathfrak{S}_n is often called the *standard representation*. It can be described as the \mathfrak{S}_n -stable complement of the subspace spanned by e_1 inside the representation $R_1 = \langle x_1, x_2, \dots, x_n \rangle$. The polynomials $x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n$ give an explicit basis of the complement.

Let $\mathfrak{m} = (x_1, x_2, \dots, x_n)$ be the irrelevant maximal ideal of R . In this section, we study homogeneous regular sequences $f_1, f_2, \dots, f_t \in R$ such that the ideal $I = (f_1, f_2, \dots, f_t)$ is stable under the action of \mathfrak{S}_n and $I/\mathfrak{m}I$ contains a copy of the standard representation. As shown in [10, Prop. 2.5], there are two possibilities: $I/\mathfrak{m}I \cong S^{(n-1,1)}$ or $I/\mathfrak{m}I \cong S^{(n-1,1)} \oplus S^{(n)}$, where $S^{(n)}$ is the one-dimensional trivial representation.

3.1. Regular sequences of type $S^{(n-1,1)}$. Here we prove the existence of regular sequences of type $S^{(n-1,1)}$ in every positive degree.

Let $\mathcal{V}_d \subset \mathbb{A}^n$ denote the affine variety cut out by the $x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d$ and $x_n = 1$; i.e.,

$$\mathcal{V}_d = \{(z_1, z_2, \dots, z_n) \in \mathbb{A}^n : z_i^d = 1, z_n = 1\}.$$

Theorem 13. Let d be a positive integer. The polynomials $x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d$ form a regular sequence of type $S^{(n-1,1)}$.

Proof. The polynomials in question form a basis of the \mathfrak{S}_n -stable complement of the one-dimensional invariant subspace spanned by $x_1^d + x_2^d + \dots + x_n^d$ inside $\langle x_1^d, x_2^d, \dots, x_n^d \rangle$. It is clear from the comments at the beginning of the section that this complement is isomorphic to $S^{(n-1,1)}$.

To prove $x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d$ form a regular sequence, we extend it by adding the polynomial x_n^d . It is clear that the two ideals $(x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d, x_n^d)$ and $(x_1^d, x_2^d, \dots, x_{n-1}^d, x_n^d)$ are equal and that the latter is generated by a regular sequence. Thus the extended sequence, and so also the original, is a regular sequence. \square

3.2. Regular sequences of type $S^{(n-1,1)} \oplus S^{(n)}$. Let $I \subseteq R$ be an \mathfrak{S}_n -stable homogeneous ideal such that $I/\mathfrak{m}I \cong S^{(n-1,1)} \oplus S^{(n)}$. Then I admits a generating set $g_1, g_2, \dots, g_{n-1}, f$ such that:

- $\deg(g_i) = d$ for $i = 1, 2, \dots, n-1$ and the vector space spanned by g_1, g_2, \dots, g_{n-1} is a representation of \mathfrak{S}_n isomorphic to $S^{(n-1,1)}$;
- $\deg(f) = a$ and $f \in R^{\mathfrak{S}_n}$.

We are interested in understanding the possible choices of degrees d and a for which such an ideal I can be generated by a regular sequence. For simplicity, we restrict to the case $g_i = x_i^d - x_n^d$ for $i = 1, 2, \dots, n-1$. This is the instance of regular sequence described in Theorem 13. Therefore our main question becomes: when can a symmetric polynomial f of degree a be chosen so that $x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d, f$ is a regular sequence?

Definition 14. Let n, d, a be three positive integers. We say the triple (n, d, a) is *good* if there exists $f \in R_a^{\mathfrak{S}_n}$ such that $x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d, f$ is a regular sequence. Otherwise (n, d, a) is called *bad*.

Remark 15. Clearly, if (n, d, a) is good, then there exists a regular sequence of type $S^{(n-1,1)} \oplus S^{(n)}$ with $S^{(n-1,1)}$ in degree d and $S^{(n)}$ in degree a . However, the converse is not true in general. For example, the triple $(5, 6, 1)$ is bad because $x_1^6 - x_5^6, x_2^6 - x_5^6, x_3^6 - x_5^6, x_4^6 - x_5^6, e_1$ is not a regular sequence. However, if we set $g_i = \sum_{j=2}^5 e_j(x_i^{6-j} - x_5^{6-j})$ for $i = 1, 2, 3, 4$, then g_1, g_2, g_3, g_4, e_1 is a regular sequence. The assertions about these sequences of polynomials can be verified computationally using the software Macaulay2 [13], and the code provided in Appendix A.

Observe that, if $f \in R$ is homogeneous, then $x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d, f$ is a regular sequence if and only if f does not vanish on \mathcal{V}_d .

For a positive integer a , the *power sum* $\mathcal{P}_a = x_1^a + x_2^a + \dots + x_n^a$ is a homogeneous symmetric polynomial of degree a . Furthermore, given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of a , we write \mathcal{P}_λ for the symmetric polynomial $\prod_{t=1}^r \mathcal{P}_{\lambda_t}$ of degree a . The set of \mathcal{P}_λ with

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ a partition of a whose parts λ_i do not exceed n is a basis of $R_a^{\mathfrak{S}_n}$ as a complex vector space (cf. [20, Prop. 7.8.2]).

Lemma 16. The triple (n, d, a) is bad if and only if there exists a point $Q \in \mathcal{V}_d$ such that $\mathcal{P}_\lambda(Q) = 0$ for every partition $\lambda \vdash a$.

Proof. If such a point Q exists, then it is clear that (n, d, a) is bad. Suppose then that (n, d, a) is bad. Enumerate the partitions $\lambda \vdash a$ whose parts do not exceed n and denote them by $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(t)}$. Introduce the homogeneous symmetric polynomial

$$f := \sum_{i=1}^t \pi^i \mathcal{P}_{\lambda^{(i)}}$$

of degree a . Since (n, d, a) is bad, there exists $Q \in \mathcal{V}_d$ such that

$$0 = f(Q) = \sum_{i=1}^t \pi^i \mathcal{P}_{\lambda^{(i)}}(Q).$$

Since the coordinates of Q are algebraic numbers, $\mathcal{P}_{\lambda^{(i)}}(Q)$ is algebraic for all $i = 1, 2, \dots, t$. Then $f(Q) = 0$ implies $\mathcal{P}_{\lambda^{(i)}}(Q) = 0$ for all $i = 1, 2, \dots, t$ because π is transcendental. The result follows. \square

The following is an immediate consequence of Lemma 16.

Corollary 17. The triple (n, d, a) is bad if and only if there exists a point $Q \in \mathcal{V}_d$ such that $f(Q) = 0$ for every $f \in R_a^{\mathfrak{S}_n}$.

Lemma 16 suggests it might be useful to understand the vanishing of power sums at roots of unity. The following result is due to Lam and Leung [15, Thm. 5.2].

Theorem 18. Let d be a positive integer and let $\Gamma(d)$ denote the numerical semi-group generated by the prime divisors of d . Then there exist d^{th} roots of unity z_1, z_2, \dots, z_n (not necessarily distinct) such that $z_1 + z_2 + \dots + z_n = 0$ if and only if $n \in \Gamma(d)$.

Note that $\Gamma(1) := \{0\}$ here.

Corollary 19. Let a, d be positive integers and let $g := \gcd(a, d)$. Then there exist d^{th} roots of unity z_1, z_2, \dots, z_n (not necessarily distinct) such that $\mathcal{P}_a(z_1, z_2, \dots, z_n) = 0$ if and only if $n \in \Gamma(d/g)$.

Proof. Assume there exist d^{th} roots of unity z_1, z_2, \dots, z_n such that $\mathcal{P}_a(z_1, z_2, \dots, z_n) = 0$. Note that z_i^a is a $(d/g)^{\text{th}}$ root of unity. Then

$$z_1^a + z_2^a + \dots + z_n^a = \mathcal{P}_a(z_1, z_2, \dots, z_n) = 0$$

implies $n \in \Gamma(d/g)$ by Theorem 18.

Conversely, assume $n \in \Gamma(d/g)$. Then Theorem 18 implies the existence of $(d/g)^{\text{th}}$ roots of unity w_1, w_2, \dots, w_n such that $w_1 + w_2 + \dots + w_n = 0$. Since $g = \gcd(a, d)$, we have

$1 = \gcd(a, d/g)$. By Bezout's identity [16, Prop. 5.1], there exist integers u, v such that $au + (d/g)v = 1$. Note that $z_i = w_i^u$ is a d^{th} root of unity. Therefore we get

$$0 = \sum_{i=1}^n w_i = \sum_{i=1}^n w_i^{au + (d/g)v} = \sum_{i=1}^n (w_i^u)^a (w_i^{d/g})^v = \mathcal{P}_a(z_1, z_2, \dots, z_n).$$

□

Remark 20. Let ζ_d be a primitive d^{th} root of unity. The Galois group of the cyclotomic field $\mathbb{Q}(\zeta_d)$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^\times$, the group of units modulo d . An element of $(\mathbb{Z}/d\mathbb{Z})^\times$ is represented by the class of an integer s coprime to d . Let γ_s denote the corresponding Galois automorphism of $\mathbb{Q}(\zeta_d)$, which is defined by fixing \mathbb{Q} and sending ζ_d to ζ_d^s . If z is a d^{th} root of unity, then z is a power of ζ_d , therefore $\gamma_s(z) = z^s$.

Now let $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$. We have that

$$\begin{aligned} \mathcal{P}_s(Q) &= z_1^s + z_2^s + \dots + z_n^s = \gamma_s(z_1) + \gamma_s(z_2) + \dots + \gamma_s(z_n) = \\ &= \gamma_s(z_1 + z_2 + \dots + z_n) = \gamma_s(\mathcal{P}_1(Q)). \end{aligned}$$

Therefore $\mathcal{P}_s(Q) = 0$ if and only if $\mathcal{P}_1(Q) = 0$.

3.3. Numerical criteria for good and bad triples. Throughout the rest of this section (n, d, a) is intended to be a triple of positive integers. We present criteria to decide whether (n, d, a) is good or bad in the sense of Definition 14.

Proposition 21. Let $g := \gcd(a, d)$. If $n \notin \Gamma(d/g)$, then (n, d, a) is good. In particular, if $n \notin \Gamma(d)$, then (n, d, a) is good for every a .

Proof. If $n \notin \Gamma(d/g)$, then \mathcal{P}_a does not vanish on \mathcal{V}_d by Corollary 19, thus (n, d, a) is good. The second assertion follows from the fact that $\Gamma(d/g) \subseteq \Gamma(d)$ for any divisor g of d . □

Remark 22. The proof of Proposition 21 uses a power sum as the symmetric polynomial of degree a . It seems that we might be able to use Theorem 18 to handle more cases by using some other symmetric polynomial f . While it is possible that $n \in \Gamma(d)$ and $f \in R^{\mathbb{S}_n}$ is homogeneous having m terms with $m \notin \Gamma(d)$, this only happens in two cases.

The first case is $f = e_n$, the n^{th} elementary symmetric polynomial, which consists of a single term and does not vanish on \mathcal{V}_d . In particular, this shows that if n divides a , then (n, d, a) is good.

The second case is essentially when d is a power of a prime. See Corollaries 32 and 33 below. In fact, suppose two distinct primes p, q divide d , $n \geq p + q$, $n \in \Gamma(d)$ and let f be a non-constant symmetric polynomial having m terms. Then $n \geq p + q$ implies that $\binom{n}{2} \geq (p-1)(q-1)$. Thus, if $m \geq \binom{n}{2}$, then $m \geq (p-1)(q-1)$, which implies $m \in \Gamma(pq)$ (cf. [18, Thm. 2.1.1]). Since $\Gamma(pq) \subseteq \Gamma(d)$, we deduce that $m \geq \binom{n}{2}$ implies $m \in \Gamma(d)$. Therefore, if $m \notin \Gamma(d)$, then $m < \binom{n}{2}$. Since we are assuming $n \in \Gamma(d)$, this implies that $f = \lambda e_n$ for some scalar λ .

Proposition 23. Define $S := \{q : q \mid d, n \notin \Gamma(d/q)\}$. If a lies in the numerical semi-group generated by S , then the triple (n, d, a) is good.

Proof. By the hypothesis, we can write $a = \sum_{i=1}^r \lambda_i$, where $\lambda_i \in S$ for $i = 1, 2, \dots, r$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$. Then $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition of a and \mathcal{P}_λ is a symmetric polynomial of degree a .

Since $\lambda_i \in S$, we have that $\lambda_i \mid d$, hence $\gcd(\lambda_i, d) = \lambda_i$. Moreover, $n \notin \Gamma(d/\lambda_i)$. Therefore Corollary 19 implies that \mathcal{P}_{λ_i} does not vanish on \mathcal{V}_d . Since this holds for all indices $i = 1, 2, \dots, r$, we conclude that $\mathcal{P}_\lambda(Q)$ does not vanish on \mathcal{V}_d . Therefore (n, d, a) is good. \square

Remark 24. Note that $d \in S$ always. Furthermore, if $d = p_1^{b_1} p_2^{b_2} \cdots p_t^{b_t}$ is the prime factorization of d , then the set

$$\left\{ \frac{d}{p_i^{b_i}} : p_i \nmid n \right\}$$

is a subset of S .

Remark 25. Proposition 23 remains true if we use $S \cup \{n\}$ instead of S . In fact, if a lies in the numerical semi-group generated by $S \cup n$, then $a = b + cn$, where b, c are positive integers and b lies in the numerical semi-group generated by S . By the proof of Proposition 23, there exists $\lambda \vdash b$ such that \mathcal{P}_λ does not vanish on \mathcal{V}_d . At the same time, the elementary symmetric polynomial e_n does not vanish on \mathcal{V}_d . Therefore $\mathcal{P}_\lambda e_n^c$ is a homogeneous symmetric polynomial of degree a which does not vanish on \mathcal{V}_d .

Proposition 26. Suppose that $n \in \Gamma(d)$ and $a \notin \Gamma(d)$. Then (n, d, a) is bad.

Proof. Since $n \in \Gamma(d)$, there exists $Q \in \mathcal{V}_d$ such that $\mathcal{P}_1(Q) = 0$ by Theorem 18. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \vdash a$, then some part λ_t is coprime to d since $a \notin \Gamma(d)$. Hence, by Remark 20, we have $\mathcal{P}_{\lambda_t}(Q) = 0$ and thus $\mathcal{P}_\lambda(Q) = 0$. The reasoning holds for all $\lambda \vdash a$. Therefore (n, d, a) is bad by Lemma 16. \square

Proposition 27. Let $g := \gcd(d, n)$. If $g \nmid a$, then (n, d, a) is bad.

Proof. Let ω be a primitive g^{th} root of unity and define $Q = (\omega, \omega^2, \dots, \omega^n) \in \mathcal{V}_d$. Observe that $\omega^i = \omega^{i+gj}$ for all $i, j \in \mathbb{Z}$. Hence, using the auxiliary variable y , we have

$$\prod_{i=1}^n (y - \omega^i) = \left[\prod_{i=1}^g (y - \omega^i) \right]^{n/g} = (y^g - 1)^{n/g}.$$

On the other hand

$$\prod_{i=1}^n (y - \omega^i) = \sum_{j=0}^n (-1)^j e_j(Q) y^{n-j}.$$

By comparing the two expressions, we deduce that $e_j(Q) = 0$ whenever $g \nmid j$. Thus the only symmetric polynomials potentially not vanishing at Q are the ones in the subring $\mathbb{C}[e_j : g \mid j]$. Note how the degree of any element in this subring is divisible g . Since $g \nmid a$, (n, d, a) is bad by Corollary 17. \square

Proposition 28. Let $g := \gcd(d, n)$ and assume that

$$a \geq \frac{(n-g)(d-g)}{g}.$$

Then (n, d, a) is bad if and only if $g \nmid a$.

Proof. If $g \nmid a$, then the triple is bad by Proposition 27.

Assume $g \mid a$ and let $a' = a/g$, $n' = n/g$, and $d' = d/g$. The inequality in the assumption gives

$$a' = \frac{a}{g} \geq \frac{n-g}{g} \frac{d-g}{g} = (n'-1)(d'-1).$$

By [18, Thm. 2.1.1], a' belongs to the numerical semi-group generated by d' and n' . Thus we can write $a' = sd' + tn'$, for some non-negative integers s and t . Multiplying by g , we obtain $a = sd + tn$. This equality implies that the homogeneous symmetric polynomial $f := \mathcal{P}_d^s e_n^t$ has degree a . For all $Q \in \mathcal{V}_d$, we have $\mathcal{P}_d(Q) = n \neq 0$. Moreover, e_n does not vanish on \mathcal{V}_d . Therefore f does not vanish on \mathcal{V}_d and the triple (n, d, a) is good. \square

3.4. Triples and prime factors. Here we analyze the property of a triple (n, d, a) being good or bad in relation to certain prime factors of n , d , and a . We begin by developing some technical results.

Let z_1, z_2, \dots, z_n be d^{th} roots of unity and consider the point $Q = (z_1, z_2, \dots, z_n) \in \mathbb{A}^n$. For an integer v , we say that Q is v -symmetric if, given a primitive v^{th} root of unity ϵ , there exists $\tau \in \mathfrak{S}_n$ such that

$$(\epsilon z_1, \epsilon z_2, \dots, \epsilon z_n) = (z_{\tau(1)}, z_{\tau(2)}, \dots, z_{\tau(n)}).$$

In other words, Q is v -symmetric if rotating each of the complex coordinates z_i by $2\pi/v$ radians produces a point in the \mathfrak{S}_n -orbit of Q . Note that $v \mid d$ because $1 = z_{\tau(1)}^d = \epsilon^d z_1^d = \epsilon^d$ and ϵ is primitive.

Lemma 29. The point $Q \in \mathcal{V}_d \subset \mathbb{A}^n$ is v -symmetric if and only if $v \mid n$ and $e_j(Q) = 0$ for all j such that $v \nmid j$.

Proof. First suppose that Q is v -symmetric. The coordinates of Q split into orbits under the cyclic group of order v acting on the complex plane by rotation. Since $Q \in \mathcal{V}_d$, we have $z_i \neq 0$ for all i . Therefore all the above orbits have cardinality v and $v \mid n$.

Since Q is p -symmetric, there is a primitive v^{th} root of unity ϵ such that, up to reordering, we may write $z_{jv+i} = \epsilon^i \omega_j$ for $1 \leq i \leq v$, $1 \leq j \leq n/v$, and for some d^{th} roots of unity ω_j . Using the auxiliary variable y , we have

$$\begin{aligned} \sum_{j=1}^n (-1)^j e_j(Q) y^j &= \prod_{i=1}^n (y - z_i) = \prod_{j=1}^{n/v} \prod_{i=1}^v (y - \epsilon^i \omega_j) = \prod_{j=1}^{n/v} \omega_j^v \prod_{i=1}^v (y/\omega_j - \epsilon^i) \\ &= \prod_{j=1}^{n/v} \omega_j^v ((y/\omega_j)^v - 1) = \prod_{j=1}^{n/v} (y^v - \omega_j^v). \end{aligned}$$

Thus $e_j(Q) = 0$ whenever $v \nmid j$.

Conversely, suppose that $v \mid n$, $Q \in \mathcal{V}_d$ and $e_j(Q) = 0$ whenever $j \nmid v$. We have

$$\prod_{i=1}^n (y - z_i) = \sum_{j=1}^n (-1)^j e_j(Q) y^j = f(y^v),$$

where f is a polynomial in one variable. At the same time

$$\prod_{i=1}^n (y - \epsilon z_i) = \epsilon^n \prod_{i=1}^n (y/\epsilon - z_i) = \epsilon^n f((y/\epsilon)^v) = f(y^v).$$

Therefore, comparing factors, we deduce that Q is symmetric. \square

Lemma 30. Suppose $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$ is v^m -symmetric and $(z_1^{v^m}, z_2^{v^m}, \dots, z_n^{v^m})$ is v -symmetric. Then Q is v^{m+1} -symmetric.

Proof. Proceeding as in the proof of Lemma 29, Q being v^m -symmetric implies the existence of a primitive $(v^m)^{\text{th}}$ root of unity ϵ such that, up to reordering, we may write $z_{jv^m+i} = \epsilon^i \omega_j$ for $1 \leq i \leq v^m$, $1 \leq j \leq n/v^m$, and for some d^{th} roots of unity ω_j . Using the auxiliary variable y , we have

$$(1) \quad \prod_{i=1}^n (y - z_i^{v^m}) = \prod_{j=1}^{n/v^m} \prod_{i=1}^{v^m} (y - \omega_j^{v^m}) = \prod_{j=1}^{n/v^m} (y - \omega_j^{v^m})^{v^m} = \left(\prod_{j=1}^{n/v^m} (y - \omega_j^{v^m}) \right)^{v^m}.$$

Since $(z_1^{v^m}, z_2^{v^m}, \dots, z_n^{v^m})$ is v -symmetric, Lemma 29 implies

$$(2) \quad \prod_{i=1}^n (y - z_i^{v^m}) = f(y^v),$$

for some polynomial f in one variable. To only way to reconcile equations (1) and (2) is if

$$\prod_{j=1}^{n/v^m} (y - \omega_j^{v^m}) = g(y^v),$$

for some polynomial g in one variable. Therefore we must have

$$\begin{aligned} \prod_{i=1}^n (y - z_i) &= \prod_{j=1}^{n/v^m} \prod_{i=1}^{v^m} (y - z_{jv^m+i}) = \prod_{j=1}^{n/v^m} \prod_{i=1}^{v^m} (y - \epsilon^i \omega_j) = \\ &= \prod_{j=1}^{n/v^m} (y^{v^m} - \omega_j^{v^m}) = g((y^{v^m})^v) = g(y^{v^{m+1}}). \end{aligned}$$

Using Lemma 29 again, we conclude that Q is v^{m+1} -symmetric. \square

Proposition 31. Let p be prime and suppose that all points $Q \in \mathcal{V}_d \subseteq \mathbb{A}^n$ with $\mathcal{P}_1(Q) = 0$ are p -symmetric. Let $g := \gcd(d, n)$ and assume $p \mid g$. Then (n, d, a) is bad if and only if $g \nmid a$.

Proof. If $g \nmid a$, then (n, d, a) is bad by Proposition 27.

We prove the other implication by contradiction, so suppose that $g \mid a$. Let $n = p^r n'$, $d = p^s d'$ and $a = p^t a'$, where $\gcd(p, n') = \gcd(p, d') = \gcd(p, a') = 1$. Set $k = \min\{r, s\}$. Since $p^k \mid g$, the condition $g \mid a$ implies $p^k \mid a$ and therefore $k \leq t$.

The hypothesis $p \mid g$ implies $s \geq 1$; hence $p \in \Gamma(d)$. At the same time, $p \mid g$ also implies $r \geq 1$; hence $n \in \Gamma(d)$. Thus, by Theorem 18, there exists $Q \in \mathcal{V}_d \subseteq \mathbb{A}^n$ such that $\mathcal{P}_1(Q) = 0$. By the hypothesis, Q is p -symmetric. However, Q is not p^{k+1} -symmetric because either $p^{k+1} \nmid n$ or $p^{k+1} \nmid d$. Therefore there is an integer m , with $1 \leq m \leq k$, such that Q is p^m -symmetric but not p^{m+1} -symmetric.

Now suppose that $\mathcal{P}_{p^m}(Q) = 0$. Then we would have

$$\mathcal{P}_1(z_1^{p^m}, z_2^{p^m}, \dots, z_n^{p^m}) = \mathcal{P}_{p^m}(Q) = 0.$$

Our hypothesis would imply that $(z_1^{p^m}, z_2^{p^m}, \dots, z_n^{p^m})$ is p -symmetric. However, Lemma 30 would give that Q is p^{m+1} -symmetric, contradicting our choice of m . Therefore $\mathcal{P}_{p^m}(Q) \neq 0$. Thus the homogeneous polynomial $(\mathcal{P}_{p^m})^{a'p^{t-m}}$ has degree a and does not vanish at Q . We conclude that (n, d, a) is good by Corollary 17. \square

In [15], Lam and Leung consider sequences z_1, z_2, \dots, z_n with each z_i a d^{th} root of unity and whose sum is 0, in particular, points $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$ such that $\mathcal{P}_1(Q) = 0$. Corollary 3.4 of [15] shows that if $d = p^r$ is a prime power, then Q must be p -symmetric. This yields the following corollary of Proposition 31.

Corollary 32. Suppose $d = p^s$ for some prime p and positive integer s . Let $g := \gcd(d, n)$. Then (n, d, a) is bad if and only if $g \nmid a$.

Proof. If $p \mid g$, then the result follows from [15, Cor. 3.4] and Proposition 31.

Assume $p \nmid g$. In this case, $g = 1 \mid a$ so we must show that (n, d, a) is good. Note that $p \nmid g$ implies $p \nmid n$. Hence $n \notin \Gamma(d) = \langle p \rangle$. Therefore (n, d, a) is good by Proposition 21. \square

Lam and Leung also showed that if (z_1, z_2, \dots, z_n) is not p -symmetric for all primes p dividing d , then $n \geq p_1(p_2 - 1) + p_3 - p_2$, where $p_1 < p_2 < p_3$ are the three smallest primes dividing d [15, Thm. 4.8]. This yields the following corollary of Proposition 31.

Corollary 33. Suppose that at least two distinct primes divide d and that $n < p + q$ where p and q are the smallest two distinct primes dividing d . Let $g := \gcd(d, n)$. Then (n, d, a) is bad if and only if $g \nmid a$.

Proof. Let $d = p^s \prod_{i=1}^m q_i^{s_i}$ be the prime factorization of d , where $p < q_1 < q_2 < \dots < q_m$. Suppose that $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$ is such that $\mathcal{P}_1(Q) = 0$. Since

$$p(q_1 - 1) + q_2 - q_1 \geq 2(q_1 - 1) + q_2 - q_1 = (q_2 - 1) + q_1 > p + q_1 > n,$$

[15, Thm. 4.8] implies that every non-empty minimal subset $I \subset \{1, 2, \dots, n\}$ such that $\sum_{i \in I} z_i = 0$ corresponds to a v -symmetric point $(z_i : i \in I)$, where v is a prime dividing d . Moreover, v divides the cardinality of I . Clearly, we may partition $\{1, 2, \dots, n\}$ into a disjoint union $I_1 \sqcup I_2 \sqcup \dots \sqcup I_t$ of such minimal subsets. Thus $n = \#I_1 + \#I_2 + \dots + \#I_t$.

Since the cardinality of each I_j is either p or some q_i , the hypothesis $n < p + q_1$ implies we must have either $t = 1$ and $n = \#I_1 = q_i$ for some i , or else $\#I_j = p$ for all j and $n = tp$.

Thus there are two possibilities: either $n = q_i$ for some q_i , or else $n = pt$. In the former case, $q_i \mid g$ and every $Q \in \mathcal{V}_d$ with $\mathcal{P}_1(Q) = 0$ is q_i -symmetric. In the latter case, $p \mid g$ and every $Q \in \mathcal{V}_d$ with $\mathcal{P}_1(Q) = 0$ is p -symmetric. Thus the hypotheses of Proposition 31 are satisfied (either with the prime q_i or with p). \square

3.5. Generating good and bad triples. We illustrate how to obtain more good and bad triples from the ones already at our disposal.

Proposition 34. Let k be a positive integer.

- (1) If (n, d, a) is bad, then (n, kd, a) is also bad.
- (2) If (n, d, a) is bad, then (kn, d, a) is also bad.
- (3) If (n, d, a) is bad, then (kn, kd, ka) is also bad.

Proof. Suppose that (n, d, a) is bad. By Corollary 17, there is a point $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d \subset \mathbb{A}^n$ such that $f(Q) = 0$ for all $f \in R_a^{\mathbb{S}^n}$. Assertion (1) follows immediately since $\mathcal{V}_d \subset \mathcal{V}_{kd}$.

For the second assertion, choose a point $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$. Define the point $Q' = (z'_1, z'_2, \dots, z'_{kn}) \in \mathcal{V}_{kd} \subset \mathbb{A}^{kn}$ by $z'_{i+n(j-1)} := z_i$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Assume, by way of contradiction, that there exists $f' \in \mathbb{C}[x_1, x_2, \dots, x_{kn}]_a^{\mathbb{S}^{kn}}$ such that $f'(Q') \neq 0$. The polynomials \mathcal{P}_λ with λ a partition of a whose parts do not exceed kn form a basis of $\mathbb{C}[x_1, x_2, \dots, x_{kn}]_a^{\mathbb{S}^{kn}}$. Then $f'(Q') \neq 0$ implies that there exists a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \vdash a$ with $\mathcal{P}_\lambda(Q') \neq 0$. Hence $\mathcal{P}_{\lambda_t}(Q') \neq 0$ for all $t = 1, 2, \dots, r$. Since

$$\mathcal{P}_{\lambda_t}(Q') = kz_1^{\lambda_t} + kz_2^{\lambda_t} + \dots + kz_n^{\lambda_t} = k\mathcal{P}_{\lambda_t}(Q),$$

we have $\mathcal{P}_{\lambda_t}(Q) \neq 0$ for all $t = 1, 2, \dots, r$, and therefore $\mathcal{P}_\lambda(Q) \neq 0$. Because $Q \in \mathcal{V}_d$ is arbitrary, Lemma 16 shows (n, d, a) is not bad. This contradicts the assumption, thus proving (2).

Now we prove part (3). By contradiction, assume (kn, kd, ka) is not bad. Given $Q \in \mathcal{V}_d$, we will construct $f \in R_a^{\mathbb{S}^n}$ such that $f(Q) \neq 0$, which will prove (n, d, a) is not bad. Consider the primitive d^{th} root of unity $\zeta := e^{2\pi i/d}$. We have $Q = (\zeta^{b_1}, \zeta^{b_2}, \dots, \zeta^{b_n})$ for some positive integers b_1, b_2, \dots, b_n . Let $\omega := e^{2\pi i/(kd)}$; observe that ω is a $(kd)^{\text{th}}$ root of unity and $\omega^k = \zeta$. Define the point $Q' = (z'_1, z'_2, \dots, z'_{kn}) \in \mathcal{V}_{kd} \subset \mathbb{A}^{kn}$ by $z'_{k(j-1)+i} := \omega^{b_j+id}$ for $1 \leq i \leq k$ and $1 \leq j \leq n$. Since we have assumed that (kn, kd, ka) is not bad, by Lemma 16, there exists a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \vdash ka$ such that $\mathcal{P}_\lambda(Q') \neq 0$. In particular, $\mathcal{P}_{\lambda_t}(Q') \neq 0$ for all $t = 1, 2, \dots, r$.

Using the auxiliary variable y , we can write

$$\begin{aligned} \prod_{t=1}^{kn} (y - z'_t) &= \prod_{j=1}^n \prod_{i=1}^k (y - \omega^{b_j+id}) = \prod_{j=1}^n \prod_{i=1}^k \omega^{b_j} (y/\omega^{b_j} - \omega^{id}) = \\ &= \prod_{j=1}^n \omega^{kb_j} \prod_{i=1}^k (y/\omega^{b_j} - (\omega^d)^i). \end{aligned}$$

Since ω^d is a primitive k^{th} root of unity, the k elements $(\omega^d)^1, (\omega^d)^2, \dots, (\omega^d)^k$ are all the k^{th} roots of unity. Therefore we get

$$\prod_{i=1}^k (y/\omega^{b_j} - (\omega^d)^i) = (y/\omega^{b_j})^k - 1.$$

Combining the two previous equations, we obtain

$$\prod_{t=1}^{kn} (y - z'_t) = \prod_{j=1}^n \omega^{kb_j} [(y/\omega^{b_j})^k - 1] = \prod_{j=1}^n (y^k - \zeta^{b_j}).$$

On the other hand, we have

$$\prod_{t=1}^{kn} (y - z'_t) = \sum_{j=0}^{kn} (-1)^j e_j(Q') y^{kn-j}.$$

By comparing these expressions, we deduce that $e_j(Q') = 0$ whenever $k \nmid j$. This implies that every homogeneous polynomial in $\mathbb{C}[x_1, x_2, \dots, x_{kn}]^{\mathfrak{S}_{kn}}$ whose degree is not divisible by k vanishes at Q' .

Thus the above integers $\lambda_1, \lambda_2, \dots, \lambda_r$ are all divisible by k and we set $c_t := \lambda_t/k$ for all $i = 1, 2, \dots, r$. We have

$$\begin{aligned} \mathcal{P}_{\lambda_t}(Q') &= \mathcal{P}_{kc_t}(Q') = \sum_{s=1}^{kn} (z'_s)^{kc_t} = \sum_{j=1}^n \sum_{i=1}^k (\omega^{b_j+id})^{kc_t} = \sum_{i=1}^k \sum_{j=1}^n (\omega^k)^{(b_j+id)c_t} = \\ &= \sum_{i=1}^k \sum_{j=1}^n \zeta^{(b_j+id)c_t} = \sum_{i=1}^k \sum_{j=1}^n (\zeta^{b_j})^{c_t} = \sum_{i=1}^k \mathcal{P}_{c_t}(Q) = k\mathcal{P}_{c_t}(Q). \end{aligned}$$

We deduce that $\mathcal{P}_{c_t}(Q) \neq 0$ for all $i = 1, 2, \dots, r$. Define $f := \prod_{t=1}^r \mathcal{P}_{\lambda_t/k} \in R$ and observe that f is an element of $R_a^{\mathfrak{S}_n}$ with $f(Q) \neq 0$. This concludes the proof. \square

As the following example illustrates, (n, d, a) being bad does not imply that any of (n, d, ka) , (n, kd, ka) , or (kn, d, ka) is bad.

Example 35. Consider $(n, d, a) = (8, 15, 4)$. Since $8 = 5 + 3 \in \Gamma(d) = \Gamma(15)$ and $4 \notin \Gamma(d)$, we see that $(8, 15, 4)$ is bad by Proposition 26.

Let $k = 2$. The triples $(n, d, ka) = (8, 15, 8)$ and $(n, kd, ka) = (8, 30, 8)$ are good because e_8 clearly does not vanish on \mathcal{V}_{15} nor on \mathcal{V}_{30} .

Now consider the triple $(kn, d, ka) = (16, 15, 8)$. Observe that

$$S = \{q : q \mid 15, 16 \notin \Gamma(15/q)\} = \{3, 5, 15\},$$

hence the numerical semi-group $\langle 3, 5 \rangle$ generated by S contains $ka = 8$. Therefore $(16, 15, 8)$ is good by Proposition 23.

Remark 36. Consider the triple (n, d, a) and let $g := \gcd(n, d)$. By Proposition 27, (n, d, a) is bad if $g \nmid a$. Thus we suppose that $g \mid a$. By Proposition 34 (3), if (n, d, a) is good, then $(n/g, d/g, a/g)$ is also good.

Proposition 37. Let k be a positive integer.

- (1) If (n, d, a) is good, then (n, d, ka) is also good.
- (2) If (n, d, a) is good, then (n, kd, ka) is also good.

Proof. Suppose that (n, d, a) is good. This implies that there exists $f \in R_a^{\mathfrak{S}_n}$ which does not vanish on \mathcal{V}_d . Assertion (1) now follows since $f^k \in R_{ka}^{\mathfrak{S}_n}$ also does not vanish on \mathcal{V}_d .

To prove (2), define

$$f'(x_1, x_2, \dots, x_n) := f(x_1^k, x_2^k, \dots, x_n^k) \in R_{ka}^{\mathfrak{S}_n}.$$

For every point $Q' = (z_1, z_2, \dots, z_n) \in \mathcal{V}_{kd}$, the point $Q = (z_1^k, z_2^k, \dots, z_n^k)$ lies in \mathcal{V}_d ; moreover, $f'(Q') = f(Q) \neq 0$. Thus (n, kd, ka) is good. \square

As the following example illustrates, (n, d, a) being good does not imply that any of (kn, d, a) , (n, kd, a) , (kn, kd, a) , (kn, d, ka) or (kn, kd, ka) is good.

Example 38. Consider $(n, d, a) = (4, 15, 1)$. Since $n = 4 \notin \Gamma(d) = \Gamma(15) = \langle 3, 5 \rangle$, we see that $(4, 15, 1)$ is good by Proposition 23.

Now consider $k = 2$ and $(kn, d, a) = (8, 15, 1)$. Since $8 \in \Gamma(15)$ and $1 \notin \Gamma(15)$, we see that $(8, 15, 1)$ is bad by Proposition 26. The triple $(n, kd, a) = (4, 30, 1)$ is bad for similar reasons. Then $(kn, kd, a) = (8, 30, 1)$ is bad as well by Proposition 34 (2).

We claim the triple $(kn, d, ka) = (8, 15, 2)$ is also bad. Using the fact that $(8, 15, 1)$ is bad, we deduce that there exists $Q \in \mathcal{V}_{15}$ such that $\mathcal{P}_1(Q) = 0$. Since 2 and 15 are coprime, Remark 20 implies $\mathcal{P}_2(Q) = 0$. Given that \mathcal{P}_2 and $\mathcal{P}_{(1,1)} = \mathcal{P}_1^2$ form a basis of the symmetric polynomials of degree 2, their simultaneous vanishing at Q implies the claim by Lemma 16. Finally, the claim just proved, together with Proposition 34 (1), implies that $(kn, kd, ka) = (8, 30, 2)$ is bad.

4. REGULAR SEQUENCES OF TYPE $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$

Throughout this section we fix $n = 4$, so $R = \mathbb{C}[x_1, x_2, x_3, x_4]$.

As proved in [10, Prop. 2.5], there exist homogeneous regular sequences g_1, g_2, f_1, f_2 in R such that g_1, g_2 form a basis of a graded representation isomorphic to $S^{(2,2)}$ and f_1, f_2 are symmetric polynomials. If $I \subset R$ is the ideal generated by g_1, g_2, f_1, f_2 , then $I/\mathfrak{m}I$ is isomorphic to $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$. Setting $a := \deg(g_1) = \deg(g_2)$, $c := \deg(f_1)$ and $d := \deg(f_2)$, we seek the possible tuples (a, a, c, d) corresponding to regular sequences g_1, g_2, f_1, f_2 of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$.

4.1. Sequences in low degree. We recall some facts of invariant theory; more details can be found in [2, Ch. 3,4]. There is an isomorphism $R \cong R^{\mathfrak{S}_4} \otimes_{\mathbb{C}} R/(e_1, e_2, e_3, e_4)$ of graded \mathfrak{S}_4 -representations. The symmetric group acts trivially on $R^{\mathfrak{S}_4}$. On the other hand, the coinvariant algebra $R/(e_1, e_2, e_3, e_4)$ is isomorphic to the regular representation of \mathfrak{S}_4 . We worked out the graded character of $R/(e_1, e_2, e_3, e_4)$ in [10, Ex. 3.1]. In particular, $R/(e_1, e_2, e_3, e_4)$ contains two copies of the irreducible representation $S^{(2,2)}$, one in degree 2 and one in degree 4.

Let us find an explicit description of these two representations. Specht's original construction shows that the polynomials

$$(3) \quad (x_1 - x_2)(x_3 - x_4), (x_1 - x_3)(x_2 - x_4)$$

span a copy of $S^{(2,2)}$ inside the degree 2 component of R (cf. [9, §7.4, Ex. 17]). Now observe that the polynomials

$$(4) \quad (x_1^2 - x_2^2)(x_3^2 - x_4^2), (x_1^2 - x_3^2)(x_2^2 - x_4^2)$$

behave in the same way under the action of \mathfrak{S}_4 . Therefore they span a copy of $S^{(2,2)}$ inside the degree 4 component of R . Note also that the polynomials in (3) and (4) do not belong to the ideal (e_1, e_2, e_3, e_4) . Therefore their residue classes span the desired copies of $S^{(2,2)}$ inside $R/(e_1, e_2, e_3, e_4)$.

Using the isomorphism $R \cong R^{\mathfrak{S}_4} \otimes_{\mathbb{C}} R/(e_1, e_2, e_3, e_4)$ together with our construction above, we can establish the following fundamental fact: any copy of $S^{(2,2)}$ contained inside the degree a component of R is spanned by

$$(5) \quad \begin{aligned} g_1 &= h(x_1 - x_2)(x_3 - x_4) + h'(x_1^2 - x_2^2)(x_3^2 - x_4^2), \\ g_2 &= h(x_1 - x_3)(x_2 - x_4) + h'(x_1^2 - x_3^2)(x_2^2 - x_4^2) \end{aligned}$$

for some symmetric polynomials h of degree $a - 2$ and h' of degree $a - 4$.

Thus, when searching for degree tuples (a, a, c, d) corresponding to regular sequences g_1, g_2, f_1, f_2 of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$, we can assume that g_1, g_2 have the form given in equation (5).

We consider the cases where $a \leq 4$ first. Clearly we must have $a \geq 2$.

Proposition 39. Let $a = 2$ or 4 . A regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ with degree tuple (a, a, c, d) exists if and only if $d \geq 2$. If $a = 3$, then no such sequence exists.

Proof. Let $a = 2$. We form polynomials g_1, g_2 as in equation (5). By degree considerations, h is a unit and $h' = 0$. Therefore we may take

$$g_1 = (x_1 - x_2)(x_3 - x_4), g_2 = (x_1 - x_3)(x_2 - x_4).$$

Now we need symmetric polynomials f_1, f_2 such that g_1, g_2, f_1, f_2 is a regular sequence. Note that f_1, f_2 cannot both be linear, otherwise they would be scalar multiples of e_1 . However, if we assume that $d = \deg(f_2) \geq 2$, then we can write $d = 2p + 3q$, where p, q are non-negative integers, and set $f_1 := e_1^c$, $f_2 := e_2^p e_3^q$. The sequence g_1, g_2, f_1, f_2 is regular with degree tuple $(2, 2, c, d)$.

Now let $a = 4$. We need h' to be a unit. In fact, we can take $h' = 1$ and $h = e_2$; this gives

$$\begin{aligned} g_1 &= e_2(x_1 - x_2)(x_3 - x_4) + (x_1^2 - x_2^2)(x_3^2 - x_4^2), \\ g_2 &= e_2(x_1 - x_3)(x_2 - x_4) + (x_1^2 - x_3^2)(x_2^2 - x_4^2). \end{aligned}$$

Again f_1, f_2 cannot both be linear. In fact, choosing the same f_1, f_2 as before gives a regular sequence g_1, g_2, f_1, f_2 with degree tuple $(4, 4, c, d)$ for $d \geq 2$.

Finally let $a = 3$. In this case, $h' = 0$ while h is a scalar multiple of e_1 . Thus g_1, g_2 have a common factor and do not form a regular sequence. \square

4.2. Sequences with $a \geq 5$. Here we obtain general results about regular sequences g_1, g_2, f_1, f_2 of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ with degree tuple (a, a, c, d) and $a \geq 5$. We still refer to the form of g_1, g_2 given in equation (5).

Lemma 40. Let

$$\begin{aligned} h_1 &:= h + h'(x_1 + x_2)(x_3 + x_4), \\ h_2 &:= h + h'(x_1 + x_3)(x_2 + x_4), \end{aligned}$$

so that $g_1 = (x_1 - x_2)(x_3 - x_4)h_1$ and $g_2 = (x_1 - x_3)(x_2 - x_4)h_2$. The sequence g_1, g_2, f_1, f_2 is regular if and only if the sequences

- (1) h, h', f_1, f_2
- (2) $(x_1 - x_2)(x_3 - x_4), (x_1 - x_3)(x_2 - x_4), f_1, f_2$
- (3) $(x_1 - x_2)(x_3 - x_4), h_2, f_1, f_2$

are regular.

Proof. By Lemma 9, g_1, g_2, f_1, f_2 is regular if and only if

- (i) h_1, h_2, f_1, f_2
- (ii) $(x_1 - x_2)(x_3 - x_4), (x_1 - x_3)(x_2 - x_4), f_1, f_2$
- (iii) $(x_1 - x_2)(x_3 - x_4), h_2, f_1, f_2$
- (iv) $h_1, (x_1 - x_3)(x_2 - x_4), f_1, f_2$

are regular. Note that (ii) and (iii) are the same as (2) and (3) above. Moreover, the transposition $(23) \in \mathfrak{S}_4$ permutes (iii) and (iv), therefore it is enough to assume (iii) is regular. Thus the statement of the lemma will follow if we can prove that (1), (2), and (3) are regular if and only if (i), (ii), and (iii) are regular.

Let us show that if (i) is regular then (1) is regular. Since we have an equality of ideals $(h_1, h_2, f_1, f_2) = (h_1, h_2 - h_1, f_1, f_2)$ and (i) is regular, $h_1, h_2 - h_1, f_1, f_2$ is also regular. Notice that

$$(6) \quad h_2 - h_1 = h'(x_1 - x_4)(x_2 - x_3).$$

This implies that h_1, h', f_1, f_2 is regular. We deduce that (1) is regular, because of the equality $(h_1, h', f_1, f_2) = (h, h', f_1, f_2)$.

Now assume that (1) and (3) are regular and let us prove that (i) is regular. Since (1) is regular, the equality $(h, h', f_1, f_2) = (h_1, h', f_1, f_2)$ implies that h_1, h', f_1, f_2 is regular. As previously observed, (3) being regular implies (iii) and (iv) are regular. Note that

(34) $h_1 = h_1$. Therefore, applying (34) to (iv), we obtain the regular sequence $h_1, (x_1 - x_4)(x_2 - x_3), f_1, f_2$. Since both h_1, h', f_1, f_2 and $h_1, (x_1 - x_4)(x_2 - x_3), f_1, f_2$ are regular, we can multiply their second elements to obtain a new regular sequence. By equation (6), this sequence is simply $h_1, h_2 - h_1, f_1, f_2$. Finally the ideal equality $(h_1, h_2 - h_1, f_1, f_2) = (h_1, h_2, f_1, f_2)$ allows us to conclude that (i) is regular. \square

By Lemma 40, a necessary condition for g_1, g_2, f_1, f_2 to be a homogeneous regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ and degrees (a, a, c, d) is that $(a-2, a-4, c, d)$ is a regular degree sequence. In fact, we will show this condition is also sufficient when $a \geq 5$.

Proposition 41. Let $a \geq 5$. Suppose that $(a-2, a-4, c, d)$ is a regular degree sequence for $n = 4$. Then there exists a homogeneous regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ and degrees (a, a, c, d) .

Proof. First we suppose that a is even. By Proposition 2, $4! \mid (a-2)(a-4)cd$. Note that both $a-2$ and $a-4$ are even, and at least one of them is divisible by 4. Therefore it is enough to account for 3 dividing $(a-2)(a-4)cd$. Moreover, we can assume $c \geq 2$ by condition (\dagger); in particular, we can write $c = 2p + 3q$ for some positive integers p, q . Table 2 contains our choices of polynomials h, h', f_1, f_2 ; one can easily verify that, in each case, h, h', f_1, f_2 is a regular sequence (see Remark 42). The polynomials g_1, g_2 are obtained using equation (5). Using Lemma 40, we conclude that, in each case, g_1, g_2, f_1, f_2 is a regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$.

Degrees				Symmetric Polynomials			
$a-2$	$a-4$	c	d	h	h'	f_1	f_2
$4\alpha + 2$	4α	3γ	d	$e_2^{2\alpha+1}$	e_4^α	e_3^γ	e_1^d
4α	$4\alpha - 2$	3γ	d	e_4^α	$e_2^{2\alpha-1}$	e_3^γ	e_1^d
$12\alpha + 2$	12α	$2p + 3q$	d	$(e_2^3 + e_3^2)e_4^{3\alpha-1}$	$(e_2^6 + e_3^4 + e_4^3)^\alpha$	$e_2^p e_3^q$	e_1^d
12α	$12\alpha - 2$	$2p + 3q$	d	$(e_2^6 + e_3^4 + e_4^3)^\alpha$	$(e_2^3 + e_3^2)e_4^{3\alpha-2}$	$e_2^p e_3^q$	e_1^d
$6\alpha, (2 \nmid \alpha)$	$6\alpha - 2$	$2p + 3q$	d	$(e_2^3 + e_3^2)^\alpha$	$e_4^{(3\alpha-1)/2}$	$e_2^p e_3^q$	e_1^d
$6\alpha + 2, (2 \mid \alpha)$	6α	$2p + 3q$	d	$e_4^{(3\alpha+1)/2}$	$(e_2^3 + e_3^2)^\alpha$	$e_2^p e_3^q$	e_1^d

TABLE 2. a even

Next suppose that a is odd. We must have $8 \mid cd$ so, without loss of generality, we may assume that $2 \mid c$ and $4 \mid d$. Furthermore, $3 \mid (a-2)(a-4)cd$. We outline our choices of h, h', f_1, f_2 in Table 3. As before, one can verify that the corresponding sequence g_1, g_2, f_1, f_2 is regular. \square

Degrees				Symmetric Polynomials			
$a - 2$	$a - 4$	c	d	h	h'	f_1	f_2
3α	$3\alpha - 2$	2γ	4δ	e_3^α	$e_2^{(3\alpha-3)/2}e_1$	$(e_1^2 + e_2)^\gamma$	e_4^δ
$3\alpha + 2$	3α	2γ	4δ	$e_2^{(3\alpha-1)/2}e_1$	e_3^α	$(e_1^2 + e_2)^\gamma$	e_4^δ
$4\alpha - 1$	$4\alpha - 3$	6γ	4δ	$e_2^{2\alpha-2}e_3$	$e_4^{\alpha-1}e_1$	$(e_2^3 + e_3^2)^\gamma$	$(e_1^4 + e_4)^\delta$
$4\alpha - 1$	$4\alpha - 3$	2γ	12δ	$e_4^{\alpha-1}e_3$	$e_2^{2\alpha-2}e_1$	$(e_1^2 + e_2)^\gamma$	$(e_3^4 + e_4^3)^\delta$
$4\alpha + 1$	$4\alpha - 1$	6γ	4δ	$e_4^\alpha e_1$	$e_2^{2\alpha-2}e_3$	$(e_2^3 + e_3^2)^\gamma$	$(e_1^4 + e_4)^\delta$
$4\alpha + 1$	$4\alpha - 1$	2γ	12δ	$e_2^{2\alpha}e_1$	$e_4^{\alpha-1}e_3$	$(e_1^2 + e_2)^\gamma$	$(e_3^4 + e_4^3)^\delta$

TABLE 3. a odd

Remark 42. For each line in Table 2 and Table 3, one can prove that the polynomials h, h', f_1, f_2 form a regular sequence using Lemma 9 and [8, Cor. 17.8 a]. As an example, we show that the polynomials in the third line of Table 2, specifically

$$(e_2^3 + e_3^2)e_4^{3\alpha-1}, \quad (e_2^6 + e_3^4 + e_4^3)^\alpha, \quad e_2^p e_3^q, \quad e_1^d,$$

form a regular sequence.

By Lemma 9 and [8, Cor. 17.8 a], it is enough to show that the sequences

- $e_2^3 + e_3^2, e_2^6 + e_3^4 + e_4^3, e_2, e_1$
- $e_2^3 + e_3^2, e_2^6 + e_3^4 + e_4^3, e_3, e_1$
- $e_4, e_2^6 + e_3^4 + e_4^3, e_2, e_1$
- $e_4, e_2^6 + e_3^4 + e_4^3, e_3, e_1$

are regular.

Let us show the first sequence is regular. The ideal it generates is equal to $e_3^2, e_3^4 + e_4^3, e_2, e_1$, therefore it suffices to show that these generators form a regular sequence. Using [8, Cor. 17.8 a] again, it is enough to prove that $e_3, e_3^4 + e_4^3, e_2, e_1$ is regular. Because of the ideal equality $(e_3, e_3^4 + e_4^3, e_2, e_1) = (e_3, e_4^3, e_2, e_1)$, we only need to prove that e_3, e_4^3, e_2, e_1 is regular. This follows immediately from [8, Cor. 17.8 a] and the fact that the elementary symmetric polynomials form a regular sequence.

The other sequences are handled similarly.

In summary, we have the following result.

Theorem 43. There exists a regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ and degrees (a, a, c, d) if and only if

- (1) $a = 2$ or 4 and $(c, d) \neq (1, 1)$, or
- (2) $a \geq 5$ and $(a - 2, a - 4, c, d)$ is a regular degree sequence.

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APPENDIX A. MACAULAY2 CODE

We present here the Macaulay2 code used to produce the example in Remark 15.

```
needsPackage "Depth"
R=QQ[x_1..x_5]
e=apply(5,i->sum(apply(subsets(gens R,i+1),product)))
l=apply(4,i->x_(i+1)^6-x_5^6)
g=apply(4,i->sum(apply(4,j->e_(j+1)*(x_(i+1)^(4-j)-x_5^(4-j)))))
isRegularSequence(l|{e_0})
isRegularSequence(g|{e_0})
```

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